HANKEL MATRICES[1]

BY
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1. **Introduction.** A Hankel matrix is a matrix, finite or infinite, whose j, k entry is a function of $j + k$. We shall present here certain theorems whose common property is that they deal with spectral properties of Hankel matrices.

Nehari [3] has shown that an infinite Hankel matrix

$$H = (c_{j+k}), \qquad j,k = 0,1,\cdots$$

represents a bounded operator on the Hilbert space l_2 if and only if there is a bounded function ϕ such that

$$c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\theta)e^{-ij\theta}d\theta, \qquad j = 0,1,\cdots, \tag{1}$$

and that

$$\|H\| = \inf\|\phi\|_\infty$$

where the infimum is taken over all ϕ satisfying (1). In §2 we shall determine analogously the smallest closed symmetric convex set containing the numerical range of H. This will be shown to be the intersection, taken over all ϕ satisfying (1), of certain convex sets explicitly described in terms of ϕ. Unfortunately the determination of the numerical range itself, rather than its symmetrization, seems much more difficult.

§3 is mainly devoted to the study of the rate of approach to zero of the eigenvalues of positive completely continuous Hankel matrices. Formulas, valid asymptotically for small ε, are found for the number of eigenvalues greater than ε. Rather than involving the c_j directly, or the functions ϕ, the formulas involve the measure on $(-1,1)$ whose moments are the c_j. With practically no more work we are able to obtain similar formulas for integral operators on $(0,\infty)$ whose kernels depend on the sum of the arguments.

In §4 we consider finite Hankel matrices. It is assumed that the c_j are ultimately positive and well behaved and various theorems are proved concerning the limiting behavior of the eigenvalues and eigenvectors of the matrices

$$H_N = (c_{j+k}), \qquad j,k = 0,1,\cdots,N-1$$

as $N \to \infty$.

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2. **The numerical range.** The numerical range of the bounded Hankel matrix H is the set

$$W(H) = \{(H\boldsymbol{a},\boldsymbol{a}):\ \|\boldsymbol{a}\| = 1\}$$
$$= \left\{ \sum_{j,k=0}^{\infty} c_{j+k} a_k \bar{a}_j :\ \sum |a_j|^2 = 1 \right\}.$$

(Here we use bold face letters to represent vectors in l_2.) $W(H)$, like the numerical range of any operator, is a convex set [2]. Let $W_s(H)$ denote the smallest symmetric convex set containing $W(H)$. We use "co" to denote convex hull.

LEMMA 2.1.

$$W_s(H) = \mathrm{co}\left\{\frac{1}{2}(H\boldsymbol{a},\bar{\boldsymbol{a}}) + \frac{1}{2}(H\bar{\boldsymbol{a}},\boldsymbol{a}):\ \|\boldsymbol{a}\| = 1\right\}.$$

Proof. Given a vector $\boldsymbol{a}$ let us write $\boldsymbol{a} = \boldsymbol{a}' + i\boldsymbol{a}''$ where $\boldsymbol{a}'$ and $\boldsymbol{a}''$ have real components. An easy computation shows that

$$(H\boldsymbol{a},\boldsymbol{a}) = (H\boldsymbol{a}',\boldsymbol{a}') + (H\boldsymbol{a}'',\boldsymbol{a}'')$$

and

$$(H\boldsymbol{a},\bar{\boldsymbol{a}}) + (H\bar{\boldsymbol{a}},\boldsymbol{a}) = (H\boldsymbol{a}',\boldsymbol{a}') - (H\boldsymbol{a}'',\boldsymbol{a}'').$$

Denote by W_0 the set which we allege $W_s(H)$ to be. The first identity above, together with the fact that $\|\boldsymbol{a}'\|^2 + \|\boldsymbol{a}''\|^2 = \|\boldsymbol{a}\|^2$, shows that $W(H) \subset W_0$. But since W_0 is convex and symmetric this implies $W_s(H) \subset W_0$. The second identity shows that for every vector $\boldsymbol{a}$ with $\|\boldsymbol{a}\| = 1$, the number

$$\frac{1}{2}(H\boldsymbol{a},\bar{\boldsymbol{a}}) + \frac{1}{2}(H\bar{\boldsymbol{a}},\boldsymbol{a})$$

is of the form $\frac{1}{2}(z' - z'')$ with $z', z'' \in W(H)$, and so belongs to $W_s(H)$. This proves $W_0 \subset W_s(H)$.

The proof of the following lemma imitates the proof of a theorem of de Leeuw and Rudin [1] which determines the extreme points of the unit sphere of H_1. (Actually, using the theorem of de Leeuw and Rudin, the Krein-Milman theorem, and other heavy machinery, one could quickly prove a weakened version of the lemma which would be sufficient for our purposes. It seems better though to simply repeat the de Leeuw-Rudin argument in the present setting.) Recall that H_p consists of those functions in $L_p(-\pi,\pi)$ whose Fourier coefficients for negative values of the index all vanish.

LEMMA 2.2. *Every function $f \in H_1$ which has L_1 norm at most 1 is the limit, in the L_1 norm, of convex combinations of squares of H_2 functions each of L_2 norm at most* 1.

Proof. Since f is the limit in L_1 of the $(C,1)$ means of its Fourier series we may assume to begin with that $f(\theta)$ is a trigonometric polynomial $\sum a_j e^{ij\theta}$. Let

$F(z)$ denote the corresponding polynomial $\sum a_j z^j$. Suppose F has a zero at a point ζ inside the unit circle and let

$$G(z) = \frac{1 - \bar{\zeta}z}{z - \zeta} F(z).$$

Then for any complex number $\beta \neq 0$,

$$F(e^{i\theta}) = \frac{1}{4}\left(\beta \frac{e^{i\theta} - \zeta}{1 - \bar{\zeta}e^{i\theta}} + 1\right)^2 \frac{G(e^{i\theta})}{\beta} - \frac{1}{4}\left(\beta \frac{e^{i\theta} - \zeta}{1 - \bar{\zeta}e^{i\theta}} - 1\right)^2 \frac{G(e^{i\theta})}{\beta}.$$

If $|\beta| = 1$ then we have, since $|\beta(e^{i\theta} - \zeta)(1 - \bar{\zeta}e^{i\theta})^{-1}| = 1$,

$$\frac{1}{2}\left|\beta \frac{e^{i\theta} - \zeta}{1 - \bar{\zeta}^{i\theta}} \pm 1\right|^2 |G(e^{i\theta})| = \left(1 \pm \mathscr{R}\beta \frac{e^{i\theta} - \zeta}{1 - \bar{\zeta}e^{i\theta}}\right) |G(z)|.$$

($\mathscr{R}$ = real part.) Thus if β is chosen so that

$$\mathscr{R}\beta \int \frac{e^{i\theta} - \zeta}{1 - \zeta e^{i\theta}} |G(e^{i\theta})| \, d\theta = 0$$

we shall have

$$\left\| \frac{1}{2}\left(\beta \frac{e^{i\theta} - \zeta}{1 - \zeta e^{i\theta}} \pm 1\right)^2 \frac{G(e^{i\theta})}{\beta} \right\|_1 = \| G(e^{i\theta}) \|_1 \leqq 1.$$

Moreover the analytic functions

$$\frac{1}{2}\left(\beta \frac{z - \zeta}{1 - \bar{\zeta}z} \pm 1\right)^2 \frac{G(z)}{\beta}$$

have one fewer zero inside the unit circle than does F. If we continue this argument repeatedly we find that $F(z)$ is a convex combination of functions $G_i(z)$ analytic on $|z| \leqq 1$, nowhere zero on $|z| < 1$, and satisfying $\| G_i(e^{i\theta}) \|_1 \leqq 1$. Each $G_i(z)$ has a square root $(G_i(z))^{1/2}$ which is analytic in $|z| < 1$ and continuous on $|z| \leqq 1$, and $(G_i(e^{i\theta}))^{1/2}$ are the desired H_2 functions.

Given a convex set K containing 0, define

$$\omega(\alpha, K) = \sup\{r \geqq 0 \colon r\alpha \in K\}$$

for complex numbers α of absolute value one. The closure of K is uniquely determined by the function $\omega(\alpha, K)$. In the next lemma, the crucial one, we denote by Φ the class of those bounded functions ϕ which satisfy (1).

Lemma 2.3. $\omega(\alpha, W_s(H)) = \frac{1}{2}\inf_{\phi \in \Phi} \| \alpha \overline{\phi(\theta)} + \bar{\alpha}\phi(-\theta) \|_\infty$.

Proof. Given a vector $\boldsymbol{a} \in l_2$ let f denote the H_2 function whose Fourier coefficients are the components of $\bar{\boldsymbol{a}}$. We have then

$$\mathscr{R}\bar{\alpha}[(H\boldsymbol{a},\bar{\boldsymbol{a}}) + (H\bar{\boldsymbol{a}},\boldsymbol{a})] = \mathscr{R}\bar{\alpha}[\sum c_{j+k}a_k a_j + \sum c_{j+k}\bar{a}_k\bar{a}_j]$$

$$= \mathscr{R} \sum (\alpha\bar{c}_{j+k} + \bar{\alpha}c_{j+k})\bar{a}_k\bar{a}_j$$

$$= \mathscr{R}\frac{1}{2\pi}\int [\alpha\overline{\phi(\theta)} + \bar{\alpha}\phi(-\theta)]f(\theta)^2 d\theta.$$

Since $\|f\|_2^2 = 2\pi\|\boldsymbol{a}\|^2$ we conclude, using Lemma 2.1, that

$$\omega(\alpha, W_s(H)) = \sup \mathscr{R}\,\bar{\alpha}W_s(H)$$

$$= \frac{1}{2}\sup\left\{\mathscr{R}\int [\alpha\overline{\phi(\theta)+}\bar{\alpha}\phi(-\theta)]f(\theta)^2 d\theta : f\in H_2, \|f\|_2 \leqq 1\right\}$$

and by Lemma 2.2 this is

$$\frac{1}{2}\sup\left\{\mathscr{R}\int [\alpha\overline{\phi(\theta)+}\bar{\alpha}\phi(-\theta)]f(\theta)d\theta : f\in H_1, \ \|f\|_1 \leqq 1\right\}.$$

Since this holds for all $\phi\in\Phi$ it is immediate that

$$\omega(\alpha, W_s(H)) \leqq \frac{1}{2}\inf_{\phi\in\Phi}\|\alpha\overline{\phi(\theta)+}\bar{\alpha}\phi(-\theta)\|_\infty.$$

To prove the opposite inequality take any $\phi\in\Phi$ and consider the linear functional on H_1, considered as a real normed linear space, given by

$$f \to \frac{1}{2}\mathscr{R}\int [\alpha\overline{\phi(\theta)} + \bar{\alpha}\phi(-\theta)]f(\theta)d\theta.$$

The norm of this linear functional is at most $\omega(\alpha, W_s(H))$, and by the Hahn-Banach theorem we can find an extension of it to L_1 which has no larger norm. Thus there is a function $\psi\in L_\infty$ such that

$$\|\psi\|_\infty \leqq \omega(\alpha, W_s(H)),$$

$$\frac{1}{2}\mathscr{R}\int [\alpha\overline{\phi(\theta)} + \bar{\alpha}\phi(-\theta)]f(\theta)d\theta = \mathscr{R}\int \psi(\theta)f(\theta)d\theta, \qquad f\in H_1.$$

This last identity shows that the Fourier coefficients of

$$\eta(\theta) = \psi(\theta) - \frac{1}{2}[\alpha\overline{\phi(\theta)} + \bar{\alpha}\phi(-\theta)]$$

for nonpositive values of the index all vanish. Thus $\phi_1 = \phi + \alpha\bar{\eta}$ belongs to Φ and

$$\frac{1}{2}\|\alpha\overline{\phi_1(\theta)} + \bar{\alpha}\phi_1(-\phi)\|_\infty = \frac{1}{2}\|\psi(\theta) + \overline{\psi(-\theta)}\|_\infty \leqq \omega(\alpha, W_s(H)).$$

This completes the proof of the lemma.

Given a pair of complex numbers z, z' denote by $E_{z,z'}$ the ellipse, having foci $\pm(zz')^{1/2}$ and major axis length $|z| + |z'|$, together with its interior. Since

$$E_{z,z'} = \left\{\frac{1}{2}(\bar{\beta}z + \beta z') : |\beta| \leqq 1\right\},$$

the set of real parts of quantities in $\bar{\alpha}E_{z,z'}$ is

$$\left\{\mathscr{R}\frac{1}{2}(\overline{\alpha\beta}z + \bar{\alpha}\beta z') : |\beta| \leqq 1\right\} = \left\{\mathscr{R}\frac{1}{2}\beta(\alpha\bar{z} + \bar{\alpha}z') : |\beta| \leqq 1\right\}.$$

Thus

$$\omega(\alpha, E_{z,z'}) = \sup \mathscr{R}\,\bar{\alpha}E_{z,z'} = \tfrac{1}{2}|\alpha\bar{z} + \bar{\alpha}z'|.$$

In view of Lemma 2.3 this suggests that the closure of $W_s(H)$ is the intersection, over all $\phi \in \Phi$, of the smallest closed convex set containing all $E_{\phi(\theta),\phi(-\theta)}$. This would be true were it not for the fact that the functions ϕ may be discontinuous. We must somehow use only those pairs $\phi(\theta), \phi(-\theta)$ which are important, and this is the reason for the following considerations.

Given a complex valued function f the essential range of f, here denoted by $R(f)$, is defined as usual to be the set of complex numbers z with the property that for each $\varepsilon > 0$ the set

$$\{\theta : |f(\theta) - z| < \varepsilon\}$$

has positive measure. Analogously given two functions f, g we define $R(f,g)$ to be the set of pairs of complex numbers (z, z') with the property that for each $\varepsilon > 0$ the set

$$\{\theta : |f(\theta) - z| < \varepsilon \text{ and } |g(\theta) - z'| < \varepsilon\}$$

has positive measure.

Lemma 2.4. *If f and g are bounded and F is a continuous function of two variables then $R(F(f,g)) = F(R(f,g))$.*

Proof. The inclusion $R(F(f,g)) \supset F(R(f,g))$ follows trivially from the definitions and the continuity of F. To prove the opposite inclusion, let $\lambda \in R(F(f,g))$. Let P and Q be closed squares, say both of side s, covering the ranges of f and g respectively. Divide P into four closed squares P_1, P_2, P_3, P_4 of side $s/2$ and Q similarly into closed squares Q_1, Q_2, Q_3, Q_4. Then for the same pair i, j $(1 \leqq i \leqq 4,\ 1 \leqq j \leqq 4)$ the set

$$\{\theta : f(\theta) \in P_i, g(\theta) \in Q_j, \text{ and } |F(f(\theta), g(\theta)) - \lambda| < \varepsilon\}$$

will have positive measure for all $\varepsilon > 0$. (One need only find a pair for which this is true for infinitely many ε's of a sequence tending to zero.) Write $P^{(1)}$ for this P_i and $Q^{(1)}$ for this Q_j. If we apply this process repeatedly we get a decreasing sequence of closed squares $P^{(k)}$ and $Q^{(k)}$ with side $s/2^k$ such that for each k the set

$$\{\theta : f(\theta) \in P^{(k)}, g(\theta) \in Q^{(k)}, \text{ and } |F(f(\theta), g(\theta)) - \lambda| < \varepsilon\}$$

has positive measure for all $\varepsilon > 0$. Let z be the common point to all the $P^{(k)}$ and z' the point common to all the $Q^{(k)}$. Then since

$$\{\theta : f(\theta) \in P^{(k)}, g(\theta) \in Q^{(k)}\}$$

has positive measure for each k we have $(z, z') \in R(f, g)$. Moreover for each ε, λ is within ε of $\bigcap_k F(P^{(k)}, Q^{(k)}) = F(z, z')$. Therefore $\lambda = F(z, z') \in F(R(f, g))$.

THEOREM 2.1. *The closure of $W_s(H)$ is*

$$\bigcap_{\phi \in \Phi} \text{co} \bigcup_{(z,z') \in R(\phi(\theta), \phi(-\theta))} E_{z,z'}.$$

Proof. Let us write

$$E_\phi = \text{co} \bigcup_{(z,z') \in R(\phi(\theta), \phi(-\theta))} E_{z,z'}.$$

By Lemma 2.3 it suffices to prove

$$\omega(\alpha, E_\phi) = \frac{1}{2} \| \alpha \overline{\phi(\theta)} + \bar{\alpha} \phi(-\theta) \|_\infty.$$

If we apply the preceding lemma with

$$f(\theta) = \phi(\theta), \; g(\theta) = \phi(-\theta), \; F(z, z') = \omega(\alpha, E_{z,z'}),$$

we obtain

$$R(\omega(\alpha, E_{\phi(0), \phi(-\theta)})) = \omega(\alpha, E_\phi).$$

Since, as we have already seen, $\omega(\alpha, E_{z,z'}) = \frac{1}{2} |\alpha \bar{z} + \bar{\alpha} z'|$, the desired identity is established.

3. **Positive Hankel matrices.** In order for the infinite Hankel matrix $H = (c_{j+k})$ to be positive, i.e. in order that every sum

$$\sum_{j,k=0}^{N} c_{j+k} a_k \bar{a}_j$$

be non-negative, it is necessary and sufficient that there exist a nondecreasing function μ on $(-\infty, \infty)$ such that

$$c_j = \int_{-\infty}^{\infty} x^j \, d\mu(x), \qquad j = 0, 1, 2, \cdots.$$

(See, for example, Theorem 1.2 of [5].) Since in order for H to represent a bounded operator on l_2 it is necessary that the c_j be bounded, it is necessary also that $\mu(x)$ be constant for $x < -1$ and for $x > 1$. Therefore throughout this section we shall assume

$$c_j = \int_{-1}^{1} x^j \, d\mu(x), \qquad j = 0,1,2,\cdots$$

where μ is a nondecreasing function on $[-1,1]$.

THEOREM 3.1. *The following are equivalent*:

(a) *H represents a bounded operator on l_2*,

(b) *$c_j = O(j^{-1})$ as $j \to \infty$*,

(c) *$\mu(1) - \mu(x) = O(1-x)$ as $x \to 1$ and $\mu(x) - \mu(-1) = O(1+x)$ as $x \to -1$.*

Proof. First we prove the equivalence of (b) and (c). If we write $\mu = \mu_1 + \mu_2$ where μ_1 is constant for $x < 0$ and μ_2 constant for $x > 0$, we see that we may assume to begin with that μ is constant for $x < 0$. Thus

$$c_j = \int_0^1 x^j \, d\mu(j) \geqq (1-\delta)^j(\mu(1) - \mu(1-\delta)).$$

If we set $j = [\delta^{-1}]$ we can conclude that (b) implies (c). Conversely if (c) holds then integration by parts gives

$$c_j = j\int_0^1 x^{j-1}[\mu(1) - \mu(x)]\, dx \leqq Aj\int_0^1 x^{j-1}(1-x)\, dx = A/(j+1)$$

for some constant A, so (b) holds.

Hilbert's inequality

$$\left| \sum_{j,k=0}^{N} \frac{a_j \bar{a}_k}{j+k+1} \right| \leqq \pi \sum_{j=0}^{N} |a_j|^2$$

shows that (b) implies (a). We shall show that (a) implies (c). In fact since

$$\sum c_{j+k} a_k \bar{a}_j = \int_{-1}^{1} \left| \sum a_j x^j \right|^2 d\mu(x)$$

as long as only finitely many a_j are not zero, we have if H is bounded

$$\int_{-1}^{1} \left| \sum a_j x^j \right|^2 d\mu(x) \leqq \|H\| \sum |a_j|^2,$$

and this holds whenever the power series is uniformly convergent on $[-1,1]$. If $a_j = r^j$ $(0 < r < 1)$ we conclude

$$\int_{-1}^{1} \frac{d\mu(x)}{(1-rx)^2} \leqq \frac{\|H\|}{1-r^2}$$

and so

$$\frac{\mu(1)-\mu(r)}{(1-r^2)^2} \leqq \int_r^1 \frac{d\mu(x)}{(1-rx)^2} \leqq \frac{\|H\|}{1-r^2},$$

$$\mu(1)-\mu(r) \leqq 2\|H\|\,(1-r),$$

and a similar inequality holds for $\mu(-r)-\mu(-1)$.

The next lemma, which we state in a general Hilbert space setting, will enable us to pass from H to a more convenient (integral) operator with similar spectral properties.

LEMMA 3.1. *Let $\mathscr{H}$ and $\tilde{\mathscr{H}}$ be Hilbert spaces and T a continuous linear transformation from $\mathscr{H}$ into $\tilde{\mathscr{H}}$ with dense range. Define the positive self-adjoint operator A on $\mathscr{H}$ by the bilinear form*

$$(Ax,y) = (Tx,Ty)$$

and define $\tilde{A}$ on $T(\mathscr{H})$ by $\tilde{A}(Tx)=T(Ax)$. Then $\tilde{A}$ is well defined, extends to a bounded positive operator on $\tilde{\mathscr{H}}$, and the spectral subspaces $\mathscr{M}_\lambda$ of A and $\tilde{\mathscr{M}}_\lambda$ of $\tilde{A}$ are related by

$$\tilde{\mathscr{M}}_\lambda = \overline{T\mathscr{M}_\lambda}$$

at every point of continuity of the spectral family of $\tilde{A}$. In addition A is completely continuous if and only if $\tilde{A}$ is.

Proof. $\tilde{A}$ is well defined since it follows from the definition of A that $Tx=Tx'$ implies $Ax=Ax'$. If $\tilde{x}=Tx$ then

$$\begin{aligned}\|\tilde{A}\tilde{x}\|^2 &= (\tilde{A}Tx,\tilde{A}Tx) = (TAx,TA\tilde{x})\\ &= (A^3x,x) \leqq \|A\|^2(Ax,x) = \|A\|^2\,\|\tilde{x}\|^2.\end{aligned}$$

(We have used the fact that $A^3 \leqq \|A\|^2A$ in the usual partial ordering of the bounded self-adjoint operators on $\mathscr{H}$.) Therefore $\tilde{A}$ extends to a bounded operator on $\tilde{\mathscr{H}}$. If $\tilde{x}=Tx$ and $\tilde{y}=Ty$ then

$$(\tilde{A}\tilde{x},\tilde{y}) = (TAx,Ty) = (A^2x,y)$$

and since A is self-adjoint and $T(\mathscr{H})$ is dense in $\tilde{\mathscr{H}}$, the extended $\tilde{A}$ is self-adjoint.

Let $\tilde{E}_\lambda$ be the spectral family of $\tilde{A}$. This may be defined by

$$\tilde{E}_\lambda = \sup\{\tilde{E}:\tilde{E}\tilde{A}=\tilde{A}\tilde{E}\leqq\lambda\tilde{E}\}$$

where the $\tilde{E}$ denote projections. At any point of continuity of $\tilde{E}_\lambda$ we also have

$$\tilde{E}_\lambda = \inf\{\tilde{E}^\perp:\tilde{E}\tilde{A}=\tilde{A}\tilde{E}\geqq\lambda\tilde{E}\}.$$

Since $\mathscr{M}_\lambda$ is invariant under A, $T\mathscr{M}_\lambda$ is invariant under $\tilde{A}$. Moreover if $\tilde{x}\in T\mathscr{M}_\lambda$, say if $\tilde{x}=Tx$ with $x\in\mathscr{M}_\lambda$, then

$$(\tilde{A}\tilde{x},\tilde{x}) = (A^2x,x) \leqq \lambda(Ax,x) = \lambda(\tilde{x},\tilde{x}).$$

This shows that $T\mathcal{M}_\lambda \subset \tilde{E}_\lambda(\tilde{\mathcal{H}}) = \tilde{\mathcal{M}}_\lambda$. Similarly $T\mathcal{M}_\lambda^\perp \subset \mathcal{M}_\lambda^\perp$. Now the subspaces $T\mathcal{M}_\lambda$ and $T\mathcal{M}_\lambda^\perp$ are mutually orthogonal, since if $x \in \mathcal{M}_\lambda$ and $y \in \mathcal{M}_\lambda^\perp$ then also $Ax \in \mathcal{M}_\lambda$ and so

$$(Tx, Ty) = (Ax, y) = 0.$$

In addition $T\mathcal{M}_\lambda + T\mathcal{M}_\lambda^\perp = T(\mathcal{M}_\lambda + \mathcal{M}_\lambda^\perp) = T(\mathcal{H})$, which is dense in $\tilde{\mathcal{H}}$. It follows that $\overline{T\mathcal{M}_\lambda}$ and $\overline{T\mathcal{M}_\lambda^\perp}$ are orthogonal complements of each other so the inclusion $T\mathcal{M}_\lambda^\perp \subset \mathcal{M}_\lambda^\perp$, which we have already seen, gives $\overline{T\mathcal{M}_\lambda} \supset \tilde{\mathcal{M}}_\lambda$. This, together with the inclusion $T\mathcal{M}_\lambda \subset \tilde{\mathcal{M}}_\lambda$, shows $\overline{T\mathcal{M}_\lambda} = \tilde{\mathcal{M}}_\lambda$.

Finally, A is completely continuous if and only if $\mathcal{M}_\lambda^\perp$ is finite dimensional for each $\lambda > 0$, and similarly for $\tilde{A}$. Since $\overline{T\mathcal{M}_\lambda^\perp} = \tilde{\mathcal{M}}_\lambda^\perp$, it is immediate that $\tilde{A}$ is completely continuous whenever A is. The converse would also be immediate if T were one-one. Thus if $\mathcal{N}$ is the null space of T and $\tilde{A}$ is completely continuous then A when restricted to $\mathcal{N}^\perp$ is completely continuous. But since A vanishes on $\mathcal{N}$, A itself is completely continuous.

To see what the lemma does for us, assume that H is a bounded Hankel matrix. Take for $\mathcal{H}$ and $\tilde{\mathcal{H}}$ the spaces l_2 and $L_2(\mu)$ respectively. Then if $\boldsymbol{a}, \boldsymbol{b} \in l_2$ we have

$$(2) \qquad (H\boldsymbol{a}, \boldsymbol{b}) = \sum_{j,k=0}^{\infty} c_{j+k} a_k b_j = \int_{-1}^{1} (\sum a_j x^j)\overline{(\sum b_j x^j)}\, d\mu(x).$$

The interchange of summation with integration is justified by Fubini's theorem since $\sum |c_{j+k} a_k \bar{b}_j| < \infty$. If we take $\boldsymbol{b} = \boldsymbol{a}$ we see that

$$T: \boldsymbol{a} \to \sum a_j x^j$$

is a continuous linear transformation from l_2 to $L_2(\mu)$. It follows from the Weierstrass approximation theorem that T has dense range. Identity (2) shows that the operator A of the lemma is just H. As for $\tilde{A}$, which we shall here call J, we have whenever $\boldsymbol{a} \in l_2$,

$$\begin{aligned} JT\boldsymbol{a} = TH\boldsymbol{a} &= \sum_j x^j \sum_k c_{j+k} a_k \\ &= \int_{-1}^{1} \sum_{j,k} x^j y^{j+k} a_k \, d\mu(y) \\ &= \int_{-1}^{1} \frac{T\boldsymbol{a}(y)}{1 - xy}\, d\mu(y). \end{aligned}$$

Now let $f \in L_2(\mu)$ be arbitrary. Then f is the limit, in $L_2(\mu)$, of a sequence of polynomials f_n. Since J is continuous,

$$Jf(x) = \lim_{n\to\infty} Jf_n(x) = \lim_{n\to\infty} \int_{-1}^{1} \frac{f_n(y)}{1 - xy}\, d\mu(y).$$

These limits are taken with respect to the norm of $L_2(\mu)$ but since

$$\lim_{n\to\infty}\int_{-1}^{1}\frac{f_n(y)}{1-xy}d\mu(y)=\int_{-1}^{1}\frac{f(y)}{1-xy}\,d\mu(y)$$

for each x in $(-1,1)$, we have almost everywhere (μ)

$$Jf(x)=\int_{-1}^{1}\frac{f(y)}{1-xy}\,d\mu(y).$$

Thus J is the integral operator on $L_2(\mu)$ with kernel $(1-xy)^{-1}$.

THEOREM 3.2. *The following are equivalent*:
(a) *H is completely continuous*,
(b) $c_j=o(j^{-1})$ *as* $j\to\infty$,
(c) $\mu(1)-\mu(x)=o(1-x)$ *as* $x\to 1$ *and* $\mu(x)-\mu(-1)=o(1+x)$ *as* $x\to -1$.

Proof. We shall not give the proof of the equivalence of (b) and (c) ,which is similar to the proof of the analogous part of Theorem 3.1.

Assume $c_j=o(j^{-1})$. Then given $\varepsilon>0$ we can write $c_j=c_j'+c_j''$ where all but finitely many c_j' are zero and where $|c_j''|\leqq\varepsilon/(j+1)$ for all j. Then $H=H'+H''$ where H' has finite rank and, by Hilbert's inequality, $\|H''\|\leqq\pi\varepsilon$. Thus H is completely continuous.

Finally assume H, and so J, is completely continuous. Let

$$f_r(x)=(\mu(1)-\mu(r))^{-1/2}\chi_{[1,r]}(x)$$

($\chi=$ characteristic function). For any $f\in L_2(\mu)$ we have by Schwarz's inequality

$$\left|\int_{-1}^{1}f_r(x)f(x)d\mu(x)\right|^2\leqq(\mu(1)-\mu(r))^{-1}\int_r^1 d\mu(x)\int_r^1|f(x)|^2d\mu(x)$$
$$=\int_r^1|f(x)|^2d\mu(x)\to 0$$

as $r\to 1$. (Note that μ is continuous at $x=1$, by Theorem 3.1.) Thus f_r converges weakly to 0, and since J is completely continuous this implies Jf_r converges strongly (i.e. in the norm of $L_2(\mu)$) to 0. Since

$$Jf_r(x)=(\mu(1)-\mu(r))^{-1/2}\int_r^1\frac{d\mu(y)}{1-xy}\geqq\frac{(\mu(1)-\mu(r))^{1/2}}{1-rx},$$

we have

$$\int_{-1}^{1}Jf_r(x)^2d\mu(x)\geqq(\mu(1)-\mu(r))\int_{-1}^{1}\frac{d\mu(x)}{(1-rx)^2}\geqq\frac{(\mu(1)-\mu(r))^2}{(1-r^2)^2}.$$

Thus $\mu(1)-\mu(r)=o(1-r)$ as $r\to 1$, and similarly $\mu(-r)-\mu(-1)=o(1-r)$.

The remainder of this section is devoted to the determination of the rate of convergence to zero of the eigenvalues of certain completely continuous H.

This is accomplished by a change of variable that transforms J into a modified convolution operator, for which there are available techniques. Suppose that μ is absolutely continuous and that f is an eigenfunction of J corresponding to the eigenvalue λ, so that

$$\int_{-1}^{1} |f(x)|^2 \mu'(x)dx < \infty, \tag{3}$$

$$\int_{-1}^{1} \frac{f(y)\mu'(y)}{1-xy}dy = \lambda f(x).$$

If we set

$$F(x) = [\mu'(\tanh x)]^{1/2} \operatorname{sech} x \, f(\tanh x)$$

then (3) is equivalent to

$$\int_{-\infty}^{\infty} |F(x)|^2 dx < \infty,$$

$$\int_{-\infty}^{\infty} \frac{[\mu'(\tanh x)\mu'(\tanh y)]^{1/2}}{\cosh (x-y)} F(y)dy = \lambda F(x).$$

Thus the eigenvalues of J, and so of H, are the eigenvalues of the integral operator on $L_2(-\infty,\infty)$ with kernel

$$[\mu'(\tanh x)\mu'(\tanh y)]^{1/2}\operatorname{sech}(x-y).$$

We shall write $\lambda_n(A)$ $(n = 0,1,2,\cdots)$ for the nth largest positive eigenvalue of a completely continuous self-adjoint operator $A; N(\varepsilon;A)$ will denote the number of eigenvalues of A which exceed ε. In both cases an eigenvalue of multiplicity m is counted m times. The minimax characterization of the eigenvalues is

$$\lambda_n(A) = \inf_{g_1,\ldots,g_n} \sup_{f\perp g_1,\ldots,g_n} \frac{(Af,f)}{(f,f)}.$$

(See, for example, [4], p. 237.) This has several well known and easy consequences which we collect together as a lemma without proof.

LEMMA 3.2. (a) *If* $A_1 \leqq A_2$ *then* $\lambda_n(A_1) \leqq \lambda_n(A_2)$ *and* $N(\varepsilon;A_1) \leqq N(\varepsilon;A_2)$.

(b) *If* B *is a self-adjoint operator satisfying* $\|B\| \leqq 1$ *then* $\lambda_n(BAB) \leqq \lambda_n(A)$ *and* $N(\varepsilon;BAB) \leqq N(\varepsilon;A)$.

(c) $\lambda_{n_1+n_2}(A_1 + A_2) \leqq \lambda_{n_1}(A_1) + \lambda_{n_2}(A_2)$ *and* $N(\varepsilon_1 + \varepsilon_2;A_1 + A_2) \leqq N(\varepsilon_1;A_1) + N(\varepsilon_2;A_2)$.

What we are going to do now is find an asymptotic formula for

$$N(\varepsilon;[V(x)\,V(y)]^{1/2}\operatorname{sech}(x-y))$$

under the assumption that V is well behaved and has support at least a half line. This will be accomplished by dividing the x,y plane into three regions, and so expressing the kernel as a sum of three kernels. All the eigenvalues of the first kernel will be small. The eigenvalues of the second kernel will be dominated by the eigenvalues of $\operatorname{sech}(x+y)$ on $(0,\infty)$ and these, we shall show are very rapidly convergent to zero. The eigenvalues of the third kernel, the main contribution, will be estimated in terms of the eigenvalues of $\operatorname{sech}(x-y)$ on a finite interval.

We shall use certain results of [7] (Lemmas 1 and 3 of §2) which can be stated as follows. Given a positive number δ, let

$$\gamma_1 = \frac{n\pi}{2(1-\delta)}, \qquad \gamma_2 = \frac{n\pi}{2(1+\delta)}.$$

Then

$$\lim_{n\to\infty} \lambda_n\left(\frac{\sin\gamma_1(x-y)}{\pi(x-y)},\ -1 \leqq x,,\ y \leqq 1\right) = 1, \tag{4}$$

and for some positive constant c_δ

$$\gamma_n\left(\frac{\sin\gamma_2(x-y)}{\pi(x-y)},\ -1 \leqq x,\ y \leqq 1\right) \leqq 1/\exp(c_\delta n) \tag{5}$$

for sufficiently large n.

Lemma 3.3. *For any $\delta > 0$ there exists constants A_δ and α_δ such that whenever $\alpha > \alpha_\delta$ we have, for all $\varepsilon > 0$,*

$$N(\varepsilon;\operatorname{sech}(x-y), 0 \leqq x,y \leqq \alpha)\begin{cases} \leqq (1+\delta)\dfrac{2\alpha}{\pi^2}\operatorname{sech}^{-1}\dfrac{1-\delta}{\pi}\varepsilon + A_\delta, \\[2ex] \geqq (1-\delta)\dfrac{2\alpha}{\pi^2}\operatorname{sech}^{-1}\dfrac{1+\delta}{\pi}\varepsilon - A_\delta, \end{cases}$$

where $\operatorname{sech}^{-1}u$ is *interpreted to be* 0 *if* $u > 1$.

Proof. First a simple change of variable gives

$$N(\varepsilon;\operatorname{sech}(x-y),\ 0 \leqq x,y \leqq \alpha) = N\left(\varepsilon;\ \frac{\alpha}{2}\operatorname{sech}\frac{\alpha}{2}(x-y),\ -1 \leqq x,y \leqq 1\right).$$

Now

$$\frac{\alpha}{2}\int_{-\infty}^{\infty} e^{i\xi x}\operatorname{sech}\frac{\alpha}{2}x\,dx = \pi\operatorname{sech}\ \frac{\pi}{\alpha}\xi$$

so that if $f \in L_2(-1,1)$ and $F(\xi) = \int_{-1}^{1} e^{i\xi x}f(x)dx$, we have

$$\int_{-1}^{1}\int_{-1}^{1}\frac{\alpha}{2}\operatorname{sech}\frac{\alpha}{2}(x-y)f(y)\overline{f(x)}\,dydx = \frac{1}{2}\int_{-\infty}^{\infty}\operatorname{sech}\frac{\pi}{\alpha}\xi\,|F(\xi)|^2d\xi$$

$$\leqq \frac{1}{2}\int_{-\gamma_2}^{\gamma_2}|F(\xi)|^2d\xi + \frac{1}{2}\operatorname{sech}\frac{\pi}{\alpha}\gamma_2\int_{-\infty}^{\infty}|F(\xi)|^2d\xi$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{\sin\gamma_2(x-y)}{\pi(x-y)}f(y)\overline{f(x)}\,dydx + \pi\operatorname{sech}\frac{\pi}{\alpha}\gamma_2\int_{-1}^{1}|f(x)|^2dx.$$

Therefore by Lemma 3.2(a) and (5)

$$\lambda_n(\operatorname{sech}(x-y),\ 0\leqq x,\ y\leqq\alpha)\leqq e^{-c\delta n}+\pi\operatorname{sech}\frac{n\pi^2}{2\alpha(1+\delta)}$$

for sufficiently large n. If we choose

$$\alpha_\delta > \frac{\pi^2}{2c_\delta(1+\delta)}$$

then we shall have for all $\alpha > \alpha_\delta$ and sufficiently large n

$$\lambda_n(\operatorname{sech}(x-y),\ 0\leqq x,\ y\leqq\alpha)\leqq\frac{\pi}{1-\delta}\operatorname{sech}\frac{n\pi^2}{2\alpha(1+\delta)},$$

which gives the first of the desired inequalities. Similarly

$$\frac{1}{2}\int_{-\infty}^{\infty}\operatorname{sech}\frac{\pi}{\alpha}\xi\,|F(\xi)|^2d\xi \geqq \frac{1}{2}\operatorname{sech}\frac{\pi}{\alpha}\gamma_1\int_{-\gamma_1}^{\gamma_1}|F(\xi)|^2d\xi$$

$$= \pi\operatorname{sech}\frac{\pi}{\alpha}\gamma_1\int_{-1}^{1}\int_{-1}^{1}\frac{\sin\gamma_1(x-y)}{\pi(x-y)}f(y)\overline{f(x)}\,dydx$$

and so, by Lemma 3.2(a) and (4) we shall have for sufficiently large n

$$\lambda_n(\operatorname{sech}(x-y),\ 0\leqq x,\ y\leqq\alpha)\geqq\frac{\pi}{1+\delta}\operatorname{sech}\frac{n\pi^2}{2\alpha(1-\delta)},$$

and this gives the other inequality.

Next we prove the needed estimate for the eigenvalues of $\operatorname{sech}(x+y)$ on $(0,\infty)$.

Lemma 3.4. *There are constants A,B such that for all $\varepsilon>0$*

$$N(\varepsilon;\operatorname{sech}(x+y),\ 0\leqq x,\ \ y<\infty)<A\log\varepsilon^{-1}+B.$$

Proof. Let $K_2(x,y)$ be the iterated kernel

$$K_2(x,y) = \int_0^{\infty}\operatorname{sech}(x+z)\operatorname{sech}(z+y)\,dz.$$

This function can be extended to be analytic in the polycylinder $|x|<\pi/2$, $|y|<\pi/2$, so we have for $0\leqq x,\ y<\pi/2$

$$K_2(x,y) = \sum_{j,k=0}^{\infty} a_{j,k}x^j y^k$$

where, for any $R > 2/\pi$,

$$|a_{j,k}| \leqq AR^{j+k}.$$

(Here and elsewhere we use the letter "A" to denote a constant, which will vary with different occurrences of the letter.) We shall take $R = 1$ for simplicity. The kernel

$$\sum_{j,k=0}^{n-1} a_{j,k}x^j y^k$$

has rank at most n, and the operator on $(0,1)$ with kernel

$$\sum_{j,k=n}^{\infty} a_{j,k}x^j y^k$$

has norm at most the square root of

$$\int_0^{1/2}\int_0^{1/2} \left|\sum_{j,k=n}^{\infty} a_{j,k}x^j y^k\right|^2 dydx \leqq A2^{-2n}.$$

Therefore by Lemma 3.2(c)

$$\lambda_n(K_2,\ 0 \leqq x,\ y \leqq \tfrac{1}{2}) \leqq A2^{-n}.$$

Next, we can write

$$K_2(x,y) = 2\sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}}{j+k+1} e^{-(2j+1)x}e^{-(2k+1)y}.$$

Suppose $f \in L_2(\tfrac{1}{2},\infty)$ and

$$\int_{1/2}^{\infty} f(x)e^{-(2j+1)x}\,dx = 0, \qquad j = 0,1,\cdots,n-1.$$

Then

$$\left|\int_{1/2}^{\infty}\int_{1/2}^{\infty} K_2(x,y)f(y)\overline{f(x)}\,dydx\right| \leqq 2\|f\|_2^2 \sum_{j,k=n}^{\infty} e^{-(j+k+1)}$$

$$\leqq Ae^{-2n}\|f\|_2^2.$$

It therefore follows from the minimax principle that

$$\lambda_n(K_2, \tfrac{1}{2} \leqq x, y < \infty) \leqq Ae^{-2n}.$$

To put together into a single estimate for $\lambda_n(K_2)$ the ones we have obtained, let P be the projection operator on $L_2(0,\infty)$ given by

$$Pf = \chi_{(1/2,\infty)} \cdot f,$$

P' the complementary projection operator

$$P'f = \chi_{(0,1/2)} \cdot f,$$

and K_2 the integral operator on $L_2(0,\infty)$ with kernel $K_2(x,y)$. Then by Lemma 3.2(c)

$$\lambda_{4n}(K_2) \leqq \lambda_{2n}(PK_2P' + P'K_2P) + \lambda_n(PK_2P) + \lambda_n(P'K_2P').$$

We have shown

$$\lambda_n(PK_2P) \leqq Ae^{-2n}, \quad \lambda_n(P'K_2P') \leqq A2^{-n}.$$

Since the eigenvalues of the square of an operator are the squares of the eigenvalues of the operator, we have

$$\begin{aligned}\lambda_{2n}(PK_2P' + P'K_2P)^2 &\leqq \lambda_{2n}(PK_2P'K_2P + P'K_2PK_2P') \\ &\leqq \lambda_n(PK_2P'K_2P) + \lambda_n(P'K_2PK_2P').\end{aligned}$$

By Lemma 3.2(b)

$$\lambda_n(PK_2P'K_2P) \leqq \lambda_n(K_2P'K_2).$$

It is an easy fact that the set of nonzero eigenvalues of the product of two operators is independent of the order of the factors, so that

$$\lambda_n(K_2P'K_2) = \lambda_n(P'K_2^2P').$$

Since K_2 is positive (and it is for this reason that we have been considering K_2 rather than $\operatorname{sech}(x+y)$) we have

$$K_2^2 \leqq \| K_2 \| K_2$$

which implies

$$P'K_2^2P' \leqq \| K_2 \| P'K_2P'$$

and so by Lemma 3.2(a)

$$\lambda_n(P'K_2^2P') \leqq \| K_2 \| \lambda_n(P'K_2P') \leqq A2^{-n}.$$

Similarly

$$\lambda_n(P'K_2PK_2P') \leqq Ae^{-2n},$$

and we deduce finally

$$\lambda_{4n}(K_2) \leqq A(e^{-n} + 2^{-n/2}).$$

Since $\lambda_n(\operatorname{sech}(x+y), 0 \leqq x, y < \infty)^2 \leqq \lambda_n(K_2)$ the desired inequality is immediate.

LEMMA 3.5. *Assume $V(x)$ is locally integrable and nonnegative, monotonic for large $|x|$, and tends to zero at $\pm\infty$. Let $\sigma(\varepsilon)$ denote the measure of the set*

$$\{x : V(x) > \varepsilon/\pi\}$$

and assume that as $\varepsilon \to 0$, $\sigma(\varepsilon) \to \infty$ *and*

$$[\sigma(\varepsilon)\log\varepsilon^{-1}]^{2/3} = o\left(\int_1^\infty \frac{\sigma(\varepsilon x)}{(x^2-1)^{1/2}}\,dx\right).$$

Then as $\varepsilon \to 0$

$$N(\varepsilon; [V(x)\,V(y)]^{1/2}\operatorname{sech}(x-y)) \sim \frac{2}{\pi^2}\int_1^\infty \frac{\sigma((\varepsilon+o(\varepsilon))x)}{(x^2-1)^{1/2}}\,dx.$$

(The condition $\sigma(\varepsilon) \to \infty$ is equivalent to V having unbounded support. The other condition on σ, the ugly one, is actually very weak. It is not hard to see for example, that if σ is absolutely continuous and

$$\lim_{\varepsilon\to 0} \sigma(\varepsilon)^{-4/3}\,|\sigma'(\varepsilon)|\;\varepsilon\log\varepsilon^{-1}$$

exists, whether finitely or infinitely, then the condition holds.)

Proof. We assume first that V is bounded and monotonic in $[0,\infty)$ and in $(-\infty,0]$, and we write $V = V_1 + V_2$ where $V_2 = 0$ in $(0,\infty)$ and $V_1 = 0$ in $(-\infty,0)$. Let $\sigma_i(\varepsilon)$ $(i = 1,2)$ be the measure of the set where V_i exceeds ε/π.

Let $\varepsilon,\delta,\alpha > 0$. (Think of ε and δ as small and α as large.) Let M be the smallest integer such that $V_1(M\alpha) \leqq \varepsilon/\pi$. (If $V_1(x) > \varepsilon/\pi$ for all x take $M = 0$.) We divide the quarter plane $x, y > 0$ into $2M+1$ regions, M squares

$$Q_k = \{k\alpha < x < (k+1)\alpha,\; k\alpha < y < (k+1)\alpha\}, \qquad 0 \leqq k < M,$$

one quarter plane

$$P = \{x > M\alpha,\; y > M\alpha\}$$

and M pairs of strips

$$S_k = \{k\alpha < x < (k+1)\alpha,\; y > (k+1)\alpha\}$$
$$\cup\,\{x > (k+1)\alpha, k\alpha < y < (k+1)\alpha\}, \qquad 0 \leqq k < M.$$

Accordingly the kernel

$$K^{(1)}(x,y) = [V_1(x)\,V_1(y)]^{1/2}\operatorname{sech}(x-y)$$

is written as a sum of $2M+1$ kernels. Let us first estimate

$$N(\eta; K^{(1)}(x,y),\; S_k).$$

By the monotonicity of V_1, and Lemma 3.2(b), this number is not decreased if first $[V_1(x)V_1(y)]^{1/2}$ is replaced by $V(k\alpha)$ and then S_k replaced by the union of quarter planes

$$S'_k = \{x < (k+1)\alpha, y > (k+1)\alpha\} \cup \{x > (k+1)\alpha, y < (k+1)\alpha\}.$$

By the translation-invariant nature of $\operatorname{sech}(x-y)$,

$$N(\eta; V(k\alpha)\operatorname{sech}(x-y), S'_k)$$
$$= N(\eta; V(k\alpha)\operatorname{sech}(x-y), \{x<0, y>0\} \cup \{x>0, y<0\}).$$

Now the second iterate of the kernel

$$\operatorname{sech}(x-y)[\chi_{(-\infty,0)}(x)\chi_{(0,\infty)}(y) + \chi_{(0,\infty)}(x)\chi_{(-\infty,0)}(y)]$$

is

$$K_2(x,y)\chi_{(0,\infty)}(x)\chi_{(0,\infty)}(y) + K_2(-x,-y)\chi_{(-\infty,0)}(x)\chi_{(-\infty,0)}(y)$$

where K_2 is, as in the proof of the preceding lemma, the second iterate of $\operatorname{sech}(x+y)$ on $(0,\infty)$. Therefore using that lemma we conclude that there are constants A and B so that for all $\eta > 0$

$$N(\eta; K^{(1)}(x,y), S_k) \leqq A \log \frac{V(k\alpha)}{\eta} + B.$$

Consequently

$$\begin{aligned} N(\delta\varepsilon; K^{(1)}(x,y), \bigcup S_k) &\leqq \sum_{k=0}^{M-1} N\left(\frac{\delta\varepsilon}{M}; K^{(1)}(x,y), S_k\right) \\ &\leqq A \sum_{k=0}^{M-1} \log \frac{MV(k\alpha)}{\delta\varepsilon} + AM \\ &\leqq A \sum_{k=0}^{M-1} \log \frac{V(k\alpha)}{\varepsilon} + AM^2 \end{aligned} \tag{6}$$

where A now depends on δ.

We consider next

$$N(\eta; K^{(1)}(x,y), Q_k).$$

It follows from the monotonicity of V_1 that this number is not decreased if $[V_1(x)V_1(y)]^{1/2}$ is replaced by $V(k\alpha)$ and not increased if it is replaced by $V((k+1)\alpha)$. After having performed one of these replacements the resulting quantity is unchanged if Q_k is replaced by $0 < x < \alpha$, $0 < y < \alpha$, and so by Lemma 3.3 we have, as long as $\alpha > \alpha_\delta$,

$$N(\eta; K^{(1)}(x,y), Q_k) \begin{cases} \leqq (1+\delta)\dfrac{2\alpha}{\pi^2}\operatorname{sech}^{-1}\dfrac{(1-\delta)\varepsilon}{\pi V(k\alpha)} + A, \\ \geqq (1-\delta)\dfrac{2\alpha}{\pi^2}\operatorname{sech}^{-1}\dfrac{(1+\delta)\varepsilon}{\pi V((k+1)\alpha)} - A. \end{cases}$$

On P we have $[V(x)V(y)]^{1/2} \leqq \varepsilon/\pi$ and so the largest eigenvalue of $K^{(1)}(x,y)\chi_P(x,y)$ is at most ε/π times the norm of the operator on $(-\infty,\infty)$ with kernel $\operatorname{sech}(x-y)$, which is

$$\int_{-\infty}^{\infty} \operatorname{sech} x\, dx = \pi.$$

It follows that

$$N(\varepsilon, K^{(1)}(x,y), P) = 0.$$

Now the $M+1$ kernels $K^{(1)}\chi_{Q_K}, K^{(1)}\chi_P$ are mutually orthogonal, their sum is a direct sum, so for any η the $N(\eta)$ for the sum is the sum of the individual $N(\eta)$'s. Therefore

$$N\left(\frac{\varepsilon}{1-\delta}, K^{(1)}(x,y), \bigcup Q_k \cup P\right) \leqq (1+\delta)\frac{2\alpha}{\pi^2}\sum_{k=1}^{M-1}\operatorname{sech}^{-1}\frac{\varepsilon}{\pi V(k\alpha)} + AM$$

and

$$N\left(\frac{\varepsilon}{1+\delta}, K^{(1)}(x,y), \bigcup Q_k \cup P\right) \geqq (1-\delta)\frac{2\alpha}{\pi^2}\sum_{k=0}^{M-2}\operatorname{sech}^{-1}\frac{\varepsilon}{\pi V((k+1)\alpha)} - AM.$$

In the last sum we have dropped the term $k = M-1$ since, by the definition of M, $V(M\alpha) < \varepsilon/\pi$. If we use these inequalities together with (6) and apply Lemma 3.2(c) we find

$$N\left(\left(\frac{1}{1-\delta}+\delta\right)\varepsilon;\, K^{(1)}(x,y)\right) \leqq \left(1+\frac{A}{\alpha}\right)(1+\delta)\frac{2\alpha}{\pi^2}\sum_{k=0}^{M-1}\operatorname{sech}^{-1}\frac{\varepsilon}{\pi V(k\alpha)} + AM^2,$$

$$N\left(\left(\frac{1}{1+\delta}-\delta\right)\varepsilon; K^{(1)}(x,y)\right) \geqq \left(1-\frac{A}{\alpha}\right)(1-\delta)\frac{2\alpha}{\pi^2}\sum_{k=1}^{M-1}\operatorname{sech}^{-1}\frac{\varepsilon}{\pi V(k\alpha)} - AM^2 - A\log\frac{V(0)}{\varepsilon},$$

where we have used the fact that $\log u^{-1} \leqq \operatorname{sech}^{-1} u$. Now

$$\sum_{k=0}^{M-1}\operatorname{sech}^{-1}\frac{\varepsilon}{\pi V(k\alpha)} \leqq \operatorname{sech}^{-1}\frac{\varepsilon}{\pi V(0)} + \int_0^{\sigma_1(\varepsilon)/\alpha}\operatorname{sech}^{-1}\frac{\varepsilon}{\pi V(x\alpha)}\,dx$$

$$= \operatorname{sech}^{-1}\frac{\varepsilon}{\pi V(0)} + \frac{1}{\alpha}\int_1^{\infty}\frac{\sigma_1(\varepsilon x)}{(x^2-1)^{1/2}}\,dx.$$

The last identity holds because each of the two integrals

$$\int_0^{\sigma_1(\varepsilon)} \operatorname{sech}^{-1} \frac{\varepsilon}{\pi V(x)}\, dx, \quad \int_1^{\infty} \frac{\sigma_1(\varepsilon x)}{(x^2 - 1)^{1/2}}\, dx$$

represents the area of the set in the first quadrant where $V(x)\operatorname{sech} y > \varepsilon/\pi$. Thus, since $M \leqq \alpha^{-1}\sigma_1(\varepsilon) + 1$,

$$N\left(\left(\frac{1}{1-\delta} + \delta\right)\varepsilon; K^{(1)}(x,y)\right)$$
$$\leqq \left(1 + \frac{A}{\alpha}\right)(1+\delta)\, \frac{2}{\pi^2} \int_1^{\infty} \frac{\sigma_1(\varepsilon x)}{(x^2 - 1)^{1/2}} dx + A\left[\frac{\sigma_1(\varepsilon)}{\alpha} + 1\right]^2 + A\alpha \log \frac{1}{\varepsilon},$$

with a similar inequality holding in the opposite direction. Analogous inequalities hold for the kernel

$$K^{(2)}(x,y) = [V_2(x)\, V_2(y)]^{1/2} \operatorname{sech}(x-y).$$

Finally if we set

$$K(x,y) = [V(x)\, V(y)]^{1/2} \operatorname{sech}(x-y)$$

we have

$$|N(\eta + \eta'; K) - [N(\eta; K^{(1)}) + N(\eta; K^{(2)})]|$$
$$\leqq N(\eta'; V(0) \operatorname{sech}(x-y), \{x<0, y>0\} \cup \{x>0, y<0\})$$
$$\leqq A \log [V(0)/\eta'] + B,$$

as we have seen before. Therefore

$$N\left(\left(\frac{1}{1-\delta} + 2\delta\right)\varepsilon; K(x,y)\right)$$
$$\leqq \left(1 + \frac{A}{\alpha}\right)(1+\delta)\frac{2}{\pi^2} \int_1^{\infty} \frac{\sigma(\varepsilon x)}{(x^2 - 1)^{1/2}} dx + A\left[\frac{\sigma(\varepsilon)}{\alpha} + 1\right]^2 + A\alpha \log \frac{1}{\varepsilon}.$$

Now set

$$\alpha = \max[\delta^{-2} + \alpha_\delta, \sigma(\varepsilon)^{2/3}(\log \varepsilon^{-1})^{-1/3}].$$

If $\alpha = \delta^{-2} + \alpha_\delta$ then $\sigma(\varepsilon)^2 \leqq A \log \varepsilon^{-1}$. Since

$$\log \frac{1}{\varepsilon} = o\left(\int_1^{\infty} \frac{\sigma(\varepsilon x)}{(x^2 - 1)^{1/2}}\, dx\right)$$

(this is an easy consequence of $\sigma(\varepsilon) \to \infty$) we would have for small enough ε and δ

$$N\left(\left(\frac{1}{1-\delta} + 2\delta\right)\varepsilon; K(x,y)\right) \leqq (1 + 2\delta)\frac{2}{\pi^2}\int_1^{\infty} \frac{\sigma(\varepsilon x)}{(x^2 - 1)^{1/2}} dx.$$

If $\alpha > \delta^{-2} + \alpha_\delta$ then

$$\alpha = \sigma(\varepsilon)^{2/3}(\log \varepsilon^{-1})^{-1/3}$$

and our other assumption on $\sigma(\varepsilon)$ yields the same inequality. An analogous inequality holds in the opposite direction.

We have assumed that V was monotonic in $[0,\infty)$ and $(-\infty,0]$. If now V is an arbitrary function satisfying the conditions of the lemma we can find a function V' which agrees with V for sufficiently large x, say for $|x| > \beta$, and which is monotonic in $[0,\infty)$ and $(-\infty,0]$. We decompose the x,y plane into the square $|x| \leqq \beta$, the two quarter planes

$$\{x > \beta, y > \beta\},$$

$$\{x < -\beta, y < -\beta\}$$

and the remainder. We now use arguments quite similar to ones already used and find that

$$N\left(\left(\frac{1}{1-\delta} + 3\delta\right)\varepsilon;\ [V(x)V(y)]^{1/2}\operatorname{sech}(x-y)\right)$$
$$\leqq (1+2\delta)\,\frac{2}{\pi^2}\int_1^\infty \frac{\sigma'(\varepsilon x)}{(x^2-1)^{1/2}}\,dx + A\log\frac{1}{\varepsilon}$$

where $\sigma'(\varepsilon)$ is the measure of the set where $V'(x) > \varepsilon/\pi$. We shall not go through the details. Since $\sigma'(\varepsilon) \leqq \sigma(\varepsilon) + 2\beta$ we find that for sufficiently small δ and ε

$$N\left(\left(\frac{1}{1-\delta} + 3\delta\right)\varepsilon;\ [V(x)V(y)]^{1/2}\operatorname{sech}(x-y)\right)$$
$$\leqq (1+3\delta)\,\frac{2}{\pi^2}\int_1^\infty \frac{\sigma(\varepsilon x)}{(x^2-1)^{1/2}}\,dx.$$

The reverse inequality holds with δ replaced by $-\delta$ everywhere, and since δ is arbitrarily small the assertion of the lemma follows.

We now have immediately the following result for the eigenvalues of Hankel matrices.

THEOREM 3.3. *Assume μ is absolutely continuous and $V(x) = \mu'(\tanh x)$ satisfies the conditions of Lemma 3.5. Then*

$$N(\varepsilon; H) \sim \frac{2}{\pi^2}\int_1^\infty \frac{\sigma((\varepsilon + o(\varepsilon))x}{(x^2-1)^{1/2}}\,dx.$$

Let us take as an example the matrix

$$H = ((j+k+a)^{-\alpha-1}), \qquad j,k = 0,1,2,\cdots,$$

where $a \neq 0, -1, -2, \cdots$ and $\alpha > 0$. If $a > 0$ we have

$$\frac{1}{(j+a)^{\alpha+1}} = \frac{1}{\Gamma(\alpha+1)} \int_0^1 x^j x^{a-1} \left(\log \frac{1}{x}\right)^\alpha dx$$

and the theorem shows, after an elementary computation, that

$$\lambda_n(H) = 1/\exp[(2\alpha n)^{1/2}\pi + o(n^{1/2})].$$

If $a < 0$ but is not a negative integer then H is no longer positive, but the same asymptotic formula holds since H is a matrix of finite rank plus a matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & H' \end{pmatrix}$$

where H' is of the same form as H but with positive a.

The preceding theorem gives us no information in case μ is supported in a closed subinterval of $(-1,1)$. In this case the c_j tend to zero exponentially as $j \to \infty$ and it is easy to conclude that the same is true of $\lambda_n(H)$ as $n \to \infty$. The following theorem gives more precise information.

THEOREM 3.4. *Assume μ' is bounded, vanishes outside the interval (a,b) where $-1 < a < b < 1$, and is bounded away from zero in every closed subinterval. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\lambda_n} = \pi K\left(\frac{(1-a^2)^{1/2}(1-b^2)^{1/2}}{1-ab}\right) \Big/ K\left(\frac{b-a}{1-ab}\right)$$

where K is the complete elliptic integral of the first kind,

$$K(r) = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{-1/2} d\theta.$$

Proof. The theorem follows easily from the special case where $\mu' = \chi_{(a,b)}$. Then

$$\begin{aligned} \lambda_n(H) &= \lambda_n(\operatorname{sech}(x-y),\ \tanh a \leqq x, y \leqq \tanh b) \\ &= \lambda_n(\beta \operatorname{sech} \beta(x-y),\ -1 \leqq x,\ y \leqq 1) \end{aligned}$$

where $\beta = \frac{1}{2}(\tanh b - \tanh a)$. The conclusion follows from Theorem II of [7].

One might wonder whether it is true that the asymptotic behavior of the c_j as $j \to \infty$ determines the asymptotic behavior of $\lambda_n(H)$ as $n \to \infty$. Theorem 3.4 shows that it is not. If for example

$$c_j = \int_0^{1/2} x^j dx, \qquad c_j' = \int_{1/4}^{1/2} x^j dx$$

then clearly $c_j \sim c_j'$, but the limits

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\lambda_n}$$

are different for the two matrices. It really seems to be the behavior of the function μ, rather than the c_j, which directly affects the asymptotic behavior of the eigenvalues.

The ideas of this section can be applied equally well to integral operators on $L_2(0,\infty)$ with kernels of the form $k(s+t)$ where

$$k(s) = \int_0^\infty e^{-sx}\, d\mu(x).$$

In analogy with the discrete case one can show that the operator is bounded if and only if $\mu(x) = O(x)$ as $x \to \infty$ and $\mu(x) - \mu(0) = O(x)$ as $x \to 0$; the conditions for complete continuity are obtained by replacing the O's by o's. If the operator is bounded then the map

$$f(s) \to \int_0^\infty x^s f(s) ds$$

is continuous from $L_2(0,\infty)$ to $L_2(\mu)$, and if we apply Lemma 3.1 with $\mathscr{H} = L_2(0,\infty)$, $\tilde{\mathscr{H}} = L_2(\mu)$, and T the above map, then A is just the integral operator on $L_2(0,\infty)$ with kernel $k(s+t)$ and $\tilde{A}$ is the integral operator on $L_2(d\mu)$ with kernel $(x+y)^{-1}$. If μ is absolutely continuous a change of variable shows that this operator has the same eigenvalues as

$$[\mu'(e^{2x})\,\mu'(e^{2y})]^{1/2}\,\mathrm{sech}(x-y)$$

on $L_2(-\infty,\infty)$, and one can easily obtain analogues of Theorems 3.3 and 3.4.

4. **Finite Hankel matrices.** We shall consider here certain limiting properties of the finite Hankel matrices

$$H_N = (c_{j+k}), \qquad j,k = 0,1,\cdots,N-1.$$

The c_j will be assumed to be non-negative throughout and the type of result obtained will depend on the behavior of the c_j as $j \to \infty$. There will be four cases:

(i) $c_j = o(j^{-1})$;

(ii) $c_j \sim L(j)/j$, L slowly varying and nondecreasing;

(iii) $c_j \sim L(j)j^\alpha$, L slowly varying and $\alpha > -1$;

(iv) $\log c_j/\log j \to \infty$.

In cases (i) and (iii) it will be shown that the H_N, suitably normalized, converge in the uniform operator topology to a limiting completely continuous operator. Since this implies that the spectral family of these normalized H_N converge uniformly to the spectral family of the limiting operator [4, p. 372], the limiting behavior of the eigenvalues and eigenvectors will have been determined. In case (ii) we shall establish an asymptotic formula for the number of eigenvalues of H_N which exceed a constant times $L(N)$, and shall determine the limiting behavior of the eigenvector corresponding to the largest eigenvalue. In case (iv) we shall determine only the asymptotic behavior of the largest eigenvalue and corresponding eigenvector.

First we prove a simple lemma.

LEMMA 4.1. *Let A_N be operators on Hilbert space, A a completely continuous operator. Assume that for any uniformly bounded sequence of vectors $\{x_N\}$, $x_N \to x$ weakly implies $A_N x_N \to Ax$ strongly. Then $\| A_N - A \| \to 0$.*

Proof. If the conclusion of the theorem were false there would exist a sequence of vectors x_N with $\| x_N \| = 1$ such that

$$\| (A_N - A)x_N \| \nrightarrow 0.$$

By choosing a subsequence if necessary we may assume that x_N converges weakly to some vector x. Then by our main assumption $A_N x_N \to Ax$ strongly, and by the complete continuity of A, $Ax_N \to Ax$ strongly. Thus

$$\| (A_N - A)x_N \| \to 0,$$

which is a contradiction.

Let P_N be the projection operator on l_2 defined by

$$P_N\{a_0, \cdots, a_{N-1}, a_N, \cdots\} = \{a_0, \cdots, a_{N-1}, 0, \cdots\}.$$

If $c_j = o(j^{-1})$ then the infinite Hankel matrix H is completely continuous (Theorem 3.2) and H_N may be identified in an obvious way with $P_N H P_N$.

THEOREM 4.1. *If $c_j = o(j^{-1})$ then $P_N H P_N \to H$ in the uniform operator topology.*

Proof. If $\boldsymbol{a}_N \to \boldsymbol{a}$ weakly then $P_N \boldsymbol{a}_N \to \boldsymbol{a}$ weakly (since $P_N \to$ identity strongly) and the complete continuity of H implies that $HP_N \boldsymbol{a}_N \to H\boldsymbol{a}$ strongly. Consequently also $P_N H P_N \boldsymbol{a}_N \to H\boldsymbol{a}$ strongly and we may apply the lemma.

If λ is an eigenvalue of H_N with corresponding eigenvector $\{a_0, \cdots, a_{N-1}\}$ then λ is an eigenvalue of the integral operator on $L_2(0,1)$ with kernel

$$Nc_{[Nx]+[Ny]}$$

and has corresponding eigenfunction $a_{[Nx]}$. Moreover all eigenvalues and eigenfunctions of the kernel arise in this way and we may therefore consider the integral operator as completely equivalent to H_N. In case (iii), $(Nc_N)^{-1}$ times this operator converges uniformly.

THEOREM 4.2. *If $c_j \sim L(j)j^{\alpha}$ with L slowly varying and $\alpha > -1$ then the integral operator on $L_2(0,1)$ with kernel*

$$c_N^{-1} \, c_{[Nx]+[Ny]}$$

converges in the uniform operator topology to the operator with kernel $(x + y)^{\alpha}$.

Proof. We first show that for any $\delta > 0$ we have

$$L(Nx) < x^{-\delta} L(N) \tag{7}$$

if Nx is sufficiently large. (Throughout the argument x and y will be confined to the open interval (0.1).) We know that for sufficiently large u we shall have

$$L(u) < (1 + \delta) L(eu)$$

and so by induction

$$L(u) < (1 + \delta)^n L(e^n u).$$

If we set $n = [\log x^{-1}]$, $u = Nx$ we deduce

$$L(Nx) < x^{-\delta} L(e^{[\log x^{-1}]} Nx)$$

and so (7) holds for large Nx.

Next we show that for some constant A,

$$c_N^{-1} c_{[Nx]+[Ny]} \leqq (x + y)^{\alpha - \delta} + A \tag{8}$$

for all N, x, y.

It follows from (7) that if $N(x + y)$ is sufficiently large,

$$\frac{c_{[Nx]+[Ny]}}{c_N} \sim \left(\frac{[Nx] + [Ny]}{N}\right)^{\alpha} \frac{L([Nx] + [Ny])}{L(N)} < (x + y)^{\alpha - \delta}.$$

If $N(x + y)$ remains bounded then

$$\frac{c_{[Nx]+[Ny]}}{c_N} \leqq \frac{A}{c_N} \leqq (x + y)^{\alpha - \delta} + A.$$

Now let f_N be a sequence of functions in $L_2(0,1)$ with $\|f_N\|_2 \leqq 1$ and $f_N \to f$ weakly. It is easy to see that for each $x > 0$

$$c_N^{-1} c_{[Nx]+[Ny]} \to (x + y)^{\alpha}$$

uniformly in y. Therefore, for each $x > 0$,

$$\begin{aligned} &\int_0^1 \frac{c_{[Nx]+[Ny]}}{c_N} f_N(y)\, dy \\ &= \int_0^1 \left[\frac{c_{[Nx]+[Ny]}}{c_N} - (x + y)^{\alpha}\right] f_N(y) dy + \int_0^1 (x + y)^{\alpha} f_N(y) dy \\ &\to \int_0^1 (x + y)^{\alpha} f(y) dy. \end{aligned}$$

But from (8), using Schwarz's inequality,

$$\left| \int_0^1 \frac{c_{[Nx]+[Ny]}}{c_N} f_N(y)\, dy \right|^2 \leqq 2 \int_0^1 (x + y)^{2(\alpha - \delta)}\, dy + A.$$

If δ is chosen so that $\alpha-\delta>-1$ then the function on the right is integrable over $(0,1)$, and we can therefore use the Lebesgue dominated convergence theorem to conclude that

$$\int_0^1 c_N^{-1} c_{[Nx]+[Ny]} f_N(y)\,dy \to \int_0^1 (x+y)^\alpha f(y)\,dy$$

in $L_2(0,1)$. The theorem therefore follows from Lemma 4.1.

In case (ii) we can apply an entirely analogous argument, but since in this case $(x+y)^{-1}$ is not the kernel of a completely continuous operator Lemma 4.1 is not applicable and we cannot assert (in fact we do not have) uniform convergence of the operator with kernel $c_N^{-1}c_{[Nx]+[Ny]}$ to the operator with kernel $(x+y)^{-1}$. Of course we do have strong convergence, and this has some consequences [4, pp. 368–369] but they are of less interest than the consequences of uniform convergence. For example little can be concluded about the limiting behavior of the largest eigenvalue or its corresponding eigenfunction. We therefore use a different approach.

In the following theorem the hypothesis concerning the c_j is stronger, but not very much so, than that stated in (ii).

THEOREM 4.3. *Assume that $L(x)$ is absolutely continuous in $x \geqq 0$ and that for large x,*

$$\frac{L'(x)}{xL(x)} \downarrow 0.$$

Let $c_n = L(n)(n+1)^{-1}$. Then for any a in $0<a<1$, $N(\pi a L(N); H_N)$, the number of eigenvalues of H_N which exceed $\pi a L(N)$, is asymptotic as $N\to\infty$ to the area in the first quadrant bounded by

$$x = \log N, \quad L(e^x)\operatorname{sech}\frac{\pi}{2}y = aL(N).$$

Proof. If L is changed on a finite interval then $N(\varepsilon; H_N)$ is changed by at most a bounded amount and the same is true of the area described. Therefore we may assume that $L(x)(x+1)^{-1}$ is nonincreasing throughout $x \geqq 0$. If we denote the eigenvalues of H_N by $\lambda_{N,j}$ $(0 \leqq j < N)$ then we have for each positive integer n,

$$\sum_{j=0}^{N-1} \lambda_{N,j}^n = \sum_{j_1,\dots,j_n=0}^{N-1} \frac{L(j_n+j_1)L(j_1+j_2)\cdots L(j_{n-1}+j_n)}{(j_n+j_1+1)(j_1+j_2+1)\cdots(j_{n-1}+j_n+1)}.$$

If in the sum on the right any one of the j's is restricted to be zero the resulting sum is $O(L(N)^n)$. Therefore if we use the well-known inequalities comparing a series of monotone terms with a corresponding integral we obtain

$$\sum_{j=0}^{N-1} \lambda_{N,j}^n = \int_0^N \cdots \int_0^N \frac{L(x_n+x_1)L(x_1+x_2)\cdots L(x_{n-1}+x_n)}{(x_n+x_1+1)(x_1+x_2+1)\cdots(x_{n-1}+x_n+1)}\,dx_1\cdots dx_n + O(L(N)^n). \tag{9}$$

Set

$$M(t_1,\cdots,t_{n-1},x) = \frac{L[(1+t_1)x-1]\cdots L[(1+t_{n-1})t_{n-2}\cdots t_1x-1]\,L[(t_1\cdots t_{n-1}+1)x]}{(1+t_1\cdots t_{n-1})(t_1+1)(t_2+1)\cdots(t_{n-1}+1)}.$$

Then if we define the new variables $t_1,\cdots,t_{n-1},x$ by

$$x_k+\tfrac{1}{2} = t_1\cdots t_k x, \qquad 1\leqq k<n,$$

$$x_n+\tfrac{1}{2} = x,$$

the integral on the right side of (9) becomes

$$\int_{1/2}^{N+1/2}\frac{dx}{x}\int\cdots\int M\,dt_1\cdots dt_{n-1}, \tag{10}$$

where the inner $n-1$-fold integral is taken over the region

$$0\leqq t_1\cdots t_{k-1}x<N+\tfrac{1}{2}, \qquad 1\leqq k<n.$$

Now given $\delta>0$ there is a constant X_1 so that

$$L(yx)\leqq(1+y)^\delta L(x), \qquad y\geqq 0, x\geqq X_1.$$

Therefore if $x\geqq X_1$ we have

$$\begin{aligned} M(t_1,\cdots,t_{n-1},x) &\leqq \frac{[1+t_1]^\delta\cdots[(1+t_{n-1})t_{n-2}\cdots t_1]^\delta[t_1\cdots t_{n-1}+1]^\delta}{(1+t_1\cdots t_{n-1})(t_1+1)\cdots(t_{n-1}+1)}L(x)^n \\ &= N(t_1,\cdots,t_{n-1})L(x)^n, \end{aligned}$$

say. If δ is sufficiently small,

$$\int_0^\infty\cdots\int_0^\infty N\,dt_1\cdots dt_{n-1}<\infty.$$

Therefore given $\varepsilon>0$ there is a region R_ε whose complement (in $0\leqq t_k<\infty$) R'_ε is bounded and bounded away from the coordinate hyperplanes and such that

$$\int_R\cdots\int N\,dt_1\cdots dt_{n-1}<\varepsilon.$$

Since L is slowly varying there is an X_2 so that whenever $x\geqq X_2$, $(t_1,\cdots,t_{n-1})\in R'_\varepsilon$ we have

$$L[(1+t_1)x-1]\cdots L[(1+t_{n-1})t_{n-2}\cdots t_1x-1]L(t_1\cdots t_{n-1}+1)x]\begin{cases}\leqq(1+\varepsilon)L(x)^n,\\ \geqq(1-\varepsilon)L(x)^n.\end{cases}$$

Consequently if $X = \max(X_1, X_2)$,

$$\int_X^{N+1/2} \frac{dx}{x} \int \cdots \int M\, dt_1 \cdots dt_{n-1}$$

$$\leqq (1+\varepsilon) \int_X^{N+1/2} \frac{L(x)^n}{x} dx \int \underset{R'_\varepsilon}{\cdots} \int \frac{dt_1 \cdots dt_{n-1}}{(1 + t_1 \cdots t_{n-1})(t_1 + 1) \cdots (t_{n-1} + 1)}$$

$$+ \varepsilon \int_X^{N+1/2} \frac{L(x)^n}{x} dx.$$

Set

$$I_n = \int_0^\infty \cdots \int_0^\infty \frac{dt_1 \cdots dt_{n-1}}{(1 + t_1 \cdots t_{n-1})(t_1 + 1) \cdots (t_{n-1} + 1)}. \tag{11}$$

Then we find

$$\int_X^{N+1/2} \frac{dx}{x} \int \cdots \int M\, dt_1 \cdots dt_{n-1}$$

$$\leqq [(1+\varepsilon) I_n + \varepsilon] \int_1^{N+1/2} \frac{L(x)^n}{x} dx.$$

It is easy to see that

$$\int_{1/2}^{N} \frac{dx}{x} \int \cdots \int M\, dt_1 \cdots dt_{n-1}$$

is bounded as $N \to \infty$, and so we conclude that the integral in (10) is at most

$$[(1+\varepsilon) I_n + 2\varepsilon] \int_1^N \frac{L(x)^n}{x} dx$$

for large N. Similar reasoning shows that the integral is at least

$$[(1-\varepsilon)^2 I_n + 2\varepsilon] \int_1^N \frac{L(x)^n}{x} dx;$$

the main difference here is that R_ε must also be so chosen that the integral in (11), extended over R' rather than $0 \leqq t_k < \infty$, is at least $(1-\varepsilon) I_n$. If we use the easy facts

$$L(N)^n = o\left(\int_1^N \frac{L(x)^n}{x} dx\right),$$

$$\int_1^N \frac{L(x)}{x} dx = O\left(L(N)^{n-1} \int_1^N \frac{L(x)}{x} dx\right),$$

then we obtain, recalling (9),

$$\sum_{j=0}^{N-1} \lambda_{N,j}^n = I_n \int_1^N \frac{L(x)^n}{x}\,dx + o\left(L(N)^{n-1}\int_1^N \frac{L(x)}{x}\,dx\right). \tag{12}$$

If in the integral (11) we set

$$\begin{aligned} t_1 &= \exp[2x_1],\\ t_2 &= \exp[2(x_2 - x_1)],\\ &\vdots\\ t_{n-1} &= \exp[2(x_{n-1} - x_{n-2})] \end{aligned}$$

then we get

$$2I_n = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \operatorname{sech} x_1 \operatorname{sech}(x_2 - x_1)\cdots \operatorname{sech}(x_{n-2} - x_{n-1})\,dx_1\cdots dx_{n-1},$$

which is the value at 0 of the n-fold convolution of $\operatorname{sech} x$. This is the integral over $(-\infty,\infty)$ of the nth power of

$$\int_{-\infty}^{\infty} e^{iyx}\operatorname{sech} x\,dx = \pi \operatorname{sech}\frac{\pi}{2}y.$$

Thus

$$I_n = \frac{1}{2}\int_{-\infty}^{\infty}\left(\pi\operatorname{sech}\frac{\pi}{2}y\right)^n dy = \int_0^{\infty}\left(\pi\operatorname{sech}\frac{\pi}{2}y\right)^n dy$$

and so (12) may be written

$$\begin{aligned}\sum_{j=0}^{N-1}\lambda_{N,j}^n = &\int_1^N\int_0^{\infty}\left(\pi L(x)\operatorname{sech}\frac{\pi}{2}y\right)^n dy\,\frac{dx}{x}\\ &+ o\left(L(N)^{n-1}\int_1^N\frac{L(x)}{x}\,dx\right).\end{aligned}$$

This was proved for $n = 2,3,\cdots$ and a somewhat simpler argument shows that it holds also for $n = 1$. It follows that for any polynomial P without constant term

$$\begin{aligned}\sum_{j=0}^{N-1} P(L(N)^{-1}\lambda_{N,j}) = &\int_1^N\int_0^{\infty} P\left(\pi\frac{L(x)}{L(N)}\operatorname{sech}\frac{\pi}{2}y\right)dy\,\frac{dx}{x}\\ &+ o\left(L(N)^{-1}\int_1^N\frac{L(x)}{x}\,dx\right).\end{aligned}$$

Since by Hilbert's inequality

$$\|H_N\| \leqq \pi\max\nolimits_{N'\leqq 2(N-1)} L(N')$$

we see that the quantities $L(N)^{-1}\lambda_{N,j}$ are bounded. The same is true of $L(N)^{-1}L(x)\operatorname{sech}\frac{1}{2}\pi y$ $(1 \leqq x \leqq N)$. Given any bounded t interval, any a in $0 < a < 1$, and any ε in $0 < \varepsilon < a$ we can find polynomials P_1, P_2 without constant term so that on this interval

$$P_1(t) \leqq \chi_{(\pi a,\infty)}(t) \leqq P_2(t),$$

$$P_2(t) - P_1(t) \leqq \varepsilon|t| + \chi_{(a-\varepsilon,a+\varepsilon)}(t/\pi).$$

Therefore we have

$$\begin{aligned}\sum_{j=0}^{N-1}\chi_{(\pi a,\infty)}(L(N)^{-1}\lambda_{N,j}) \leqq & \int_1^N\int_0^\infty \chi_{(a,\infty)}\left(\frac{L(x)}{L(N)}\operatorname{sech}\frac{\pi}{2}y\right)dy\,\frac{dx}{x} \\ & + \left(\frac{\pi}{2}\varepsilon + o(1)\right)L(N)^{-1}\int_1^N \frac{L(x)}{x}dx \\ & + \frac{2}{\pi}\int_1^N\left[\operatorname{sech}^{-1}(a-\varepsilon)\frac{L(N)}{L(x)} - \operatorname{sech}^{-1}(a+\varepsilon)\frac{L(N)}{L(x)}\right]\frac{dx}{x}.\end{aligned} \tag{13}$$

The left side of the inequality is the number of eigenvalues which exceed $\pi a L(N)$ and the double integral on the right side, which we call shall I, represents the area described in the statement of the theorem.

We show next that

$$L(N)^{-1}\int_1^N \frac{L(x)}{x}dx = O(I). \tag{14}$$

If $L'(x) = 0$ for arbitrarily large x, then, by our main assumption, L is constant for sufficiently large x and both the left side of (14) and I are asymptotically nonzero constants times $\log N$. Assume therefore that L' is ultimately positive and set

$$\rho(N) = \inf\{x : L(x) > \frac{1+a}{2}L(N)\}.$$

Then

$$\begin{aligned} I &= \frac{2}{\pi}\int_1^N \operatorname{sech}^{-1}\frac{aL(N)}{L(x)}\frac{dx}{x} \\ &\geqq \frac{2}{\pi}\operatorname{sech}^{-1}\frac{2a}{1+a}\int_{\rho(N)}^N \frac{dx}{x}.\end{aligned}$$

Let A be so large that $xL'(x)L(x)^{-1}$ is nonincreasing for $x \geqq A$. Then if in the integral $\int_A^N x^{-1}L(x)dx$ we make the change of variable $y = L(x)$ we obtain

$$\begin{aligned}\int_A^N \frac{L(x)}{x}dx &= \int_{L(A)}^{L(N)} \frac{L(x)}{xL'(x)}dy \\ &\leqq 2\int_{1/2L(N)}^{L(N)} \frac{L(x)}{xL'(x)}dy \\ &= 2\int_{\rho(N)}^{N} \frac{L(x)}{x}dx \\ &\leqq 2L(N)\int_{\rho(N)}^{N} \frac{dx}{x}.\end{aligned}$$

Therefore $\int_A^N x^{-1}L(x)dx = O(IL(N))$ and (14) is established.

For the remaining integral in (13) take $\varepsilon < a/2$. Then by the monotonicity of L,

$$(a-\varepsilon)\,\frac{L(N)}{L(x)} > \frac{a}{2}.$$

Now there is a constant A with the property that if $a/2 < u < v$ then

$$\operatorname{sech}^{-1}u - \operatorname{sech}^{-1}v \leqq A(v-u)^{1/2},$$

so the integrand in the last integral of (13) is at most

$$A\left(2\,\varepsilon\,\frac{L(N)}{L(x)}\right)^{1/2}.$$

But if $(a-\varepsilon)L(N)/L(x) \geqq 1$ the integrand is zero. Therefore the integrand is at most

$$A\left(\frac{2\varepsilon}{a-\varepsilon}\right)^{1/2} \leqq 2A\left(\frac{\varepsilon}{a}\right)^{1/2}$$

and so the last term in (13) is at most

$$A\varepsilon^{1/2}\log N$$

with a different A. If we use this, the fact that $\log N = O(I)$, and (14), then we obtain from (13)

$$\sum_{j=0}^{N-1} \chi_{(\pi a,\infty)}(L(N)^{-1}\lambda_{N,j}) \leqq (1 + o(1))I.$$

The opposite inequality can be obtained similarly and so the theorem is proved.

We have the following corollary, which was conjectured by Wilf [8].

Corollary. *For the Hilbert matrix*

$$((j+k+\alpha)^{-1}), \qquad j,k = 0,1,\cdots,N-1,$$

with $\alpha \neq 0, -1, -2, \cdots$ *the number of eigenvalues which exceed* πa $(0 < a < 1)$ *is asymptotically*

$$\frac{2}{\pi} \log N \operatorname{sech}^{-1} a.$$

Let us return to a general Hankel matrix H_N in case (ii) and consider the behavior of the norm (largest eigenvalue) and corresponding eigenvector. As was mentioned in the discussion preceding the statement of Theorem 4.3, the integral operator on $L_2(0,1)$ with kernel

$$c_N^{-1} c_{[Nx]+[Ny]}$$

converges strongly to the integral operator with kernel $(x+y)^{-1}$, an operator with norm π. It follows easily that

$$\liminf_{N \to \infty} \frac{\| H_N \|}{L(N)} \geqq \pi.$$

Conversely

$$\frac{\| H_N \|}{L(N)} \leqq \pi \max_{n \leqq 2N-2} \frac{(n+1)c_n}{L(N)}.$$

If either $(n+1)c_n$ is nondecreasing for all $n \geqq 0$ or $L(N) \to \infty$ the right side will approach π as $N \to \infty$ and we shall have $\| H_N \| \sim \pi L(N)$. However if

$$L(N) \to L_\infty < \infty$$

but L is not nondecreasing for all n we may well have $\liminf \| H_N \| > \pi L_\infty$. (For example if $c_n = n^{-1}$ for $n > 0$ then $\| H_N \| \geqq c_0 - \pi$ for all N.) Nevertheless $\| H_N \|$ does approach a finite limit in this case, namely the norm of the infinite Hankel matrix H. In any case let us set

$$\lambda = \begin{cases} \pi & L \text{ unbounded}, \\ \dfrac{\| H \|}{\lim L(N)} & L \text{ bounded}. \end{cases}$$

Then $\lambda \geqq \pi$ and

$$\frac{\| H_N \|}{L(N)} \to \lambda, \qquad N \to \infty.$$

Let $\{a_{N,j}\}$ $(0 \leqq j < N)$ be the eigenvector of H_N corresponding to its largest eigenvalue $\| H_N \|$; we assume all $a_{N,j} \geqq 0$ (as we may since the matrix has non-negative entries) and the $a_{N,j}$ are normalized so that

$$\frac{1}{N} \sum_{j=0}^{N-1} a_{N,j} = 1.$$

We write

$$f_N(x) = a_{N,[Nx]} \qquad 0 \leqq x < 1.$$

THEOREM 4.4. *Let* ξ *be the smallest positive root of*

$$\sin \pi\xi = \frac{\pi}{\lambda},$$

so $0 < \xi \leqq \frac{1}{2}$. *Then* $f_N(x)$ *converges to* $f(x)$ (a) *uniformly in* $(\delta, 1)$ *for each* $\delta > 0$ *and* (b) *weakly in the sense that* $\int_0^1 f_N(x)g(x)dx \to \int_0^1 f(x)g(x)dx$ *for all continuous* g, *where*

$$f(x) = \frac{\Gamma\left(1 - \frac{\xi}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\xi}{2}\right)}{4\pi^{3/2}\lambda i} \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{\xi}{2} + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{\xi}{2} + \frac{s}{2}\right)}{\Gamma\left(1 + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)} x^{-s-1} ds.$$

Proof. The matrix equation

$$\sum_{k=0}^{N-1} c_{j+k} a_{N,k} = \| H_N \| a_{N,j}$$

is equivalent to the integral equation

$$(15) \qquad \int_0^1 c_N^{-1} c_{[Nx]+[Ny]} f_N(y)\, dy = (Nc_N)^{-1} \| H_N \| f_N(x).$$

Now to prove the theorem it suffices to prove that every sequence $\{N'\}$ has a subsequence $\{N''\}$ for which the desired convergence holds. Since the f_N are non-negative with integral one, we can find a function $\mu(x)$ nondecreasing for $0 \leqq x \leqq 1$ and a subsequence $\{N''\}$ such that

$$\int_0^1 g(x) f_{N''}(x)\, dx \to \int_0^1 g(x)\, d\mu(x)$$

for any g which is continuous on $0 \leqq x \leqq 1$. We shall henceforth write N for N''. Since for any $x > 0$, $(x+y)^{-1}$ is continuous on $0 \leqq y \leqq 1$ and

$$c_N^{-1} c_{[Nx]+[Ny]} \to (x+y)^{-1}$$

uniformly, we have

$$\int_0^1 c_N^{-1} c_{[Nx]+[Ny]} f_N(y)\, dy$$

$$= \int_0^1 \left(\frac{c_{[Nx]+[Ny]}}{c_N} - \frac{1}{x+y}\right) f_N(y)\, dy + \int_0^1 \frac{1}{x+y} f_N(y)\, dy$$

$$\to \int_0^1 \frac{1}{x+y}\, d\mu(y).$$

Since $(Nc_N)^{-1}\| H_N \| \to \lambda$ it follows from this and (15) that

$$f(x) = \lim_{N \to \infty} f_N(x)$$

exists for each $x > 0$ and that

$$\int_0^1 \frac{1}{x+y}\, d\mu(y) = \lambda f(x). \tag{16}$$

But since the f_N are uniformly bounded on every interval $(\delta, 1]$ (this follows from (15), the uniform boundedness of $c_N^{-1}c_{[Nx]+[Ny]}$, and the fact that $\|f_N\|_1 = 1$) μ is absolutely continuous on $(0,1]$ and $\mu'(x) = f(x)$ there. We can therefore write (16) as

$$\frac{\mu(0+) - \mu(0)}{x} + \int_0^1 \frac{f(y)}{x+y}\, dy = \lambda f(x).$$

But $f \in L_1$ implies that the integral on the left side is at most $o(x^{-1})$ as $x \to 0$, so that if $\mu(0+) - \mu(0) \neq 0$ we would have $\lambda f(x) \notin L_1$, which is a contradiction. Thus μ is continuous at zero and

$$\int_0^1 \frac{f(y)}{x+y}\, dy = \lambda f(x). \tag{17}$$

We know that $f_N(x) \to f(x)$ weakly. Uniform convergence on $(\delta, 1)$ follows from the facts that $f_N(x) \to f(x)$ boundedly and pointwise and that $c_N^{-1}c_{[Nx]+[Ny]}$ converges to $(x+y)^{-1}$ uniformly for $x \geqq \delta$, $0 \leqq y \leqq 1$.

It remains to prove that $f(x)$ is what it is claimed to be. Since $\int_0^1 f(x)dx = 1$ we can write (17) as

$$\int_0^1 \left(\frac{1}{x+y} - \frac{1}{x}\right) f(y)dy = \lambda f(x) - \frac{1}{x}.$$

Let this identity, which holds for $0 < x \leqq 1$, be used to define $f(x)$ for $x > 1$. Then $\lambda f(x) - x^{-1}$ is $O(x^{-1})$ for small x and $O(x^{-2})$ for large x. Let $0 < \mathscr{R}s < 1$. If we multiply both sides of the equation by x^s and integrate with respect to x over $(0, \infty)$ we obtain

$$\int_0^\infty \left(\lambda f(x) - \frac{1}{x}\right) x^s\, dx = -\int_0^1 y f(y)\, dy \int_0^\infty \frac{x^{s-1}}{x+y}\, dx$$

$$= -\pi \csc \pi s \int_0^1 f(y) y^s dy,$$

the interchange of the order of integration being justified by Fubini's theorem. Set

$$\Phi(s) = \int_0^1 f(x) x^s\, dx,$$

$$\psi(s) = \int_1^\infty \left(\lambda f(x) - \frac{1}{x}\right) x^s\, dx.$$

Then Φ is analytic in $\mathscr{R}s > 0$ and bounded in $\mathscr{R}s \geqq \delta > 0$ and Ψ is analytic is $\mathscr{R}s < 1$ and bounded in $\mathscr{R}s \leqq 1 - \delta < 1$. Moreover

$$(\lambda + \pi \csc \pi s)\Phi(s) = \frac{1}{s} - \Psi(s).$$

The next step in this complex-variable solution of (17) is what is essentially the Weiner-Hopf factorization of $\lambda + \pi \csc \pi s$. We have (recall the definition of ξ)

$$1 + \frac{\pi}{\lambda} \csc \pi s = \frac{\sin \frac{\pi}{2}(\xi + s) \cos \frac{\pi}{2}(\xi - s)}{\sin \frac{\pi}{2} s \cos \frac{\pi}{2} s}$$

$$= \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right)\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{\xi}{2} + \frac{s}{2}\right)\Gamma\left(1 - \frac{\xi}{2} - \frac{s}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\xi}{2} - \frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{\xi}{2} + \frac{s}{2}\right)}$$

so

$$\lambda \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)}{\Gamma\left(\frac{\xi}{2} + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{\xi}{2} + \frac{s}{2}\right)} \Phi(s)$$

$$= \frac{\Gamma\left(1 - \frac{\xi}{2} - \frac{s}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\xi}{2} - \frac{s}{2}\right)}{\Gamma\left(1 - \frac{s}{2}\right)\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)} \left(\frac{1}{s} - \Psi(s)\right).$$

The left side is analytic on $\mathscr{R}s \geqq \frac{1}{2}$ and is bounded as $|s| \to \infty$; the right side is analytic on $\mathscr{R}s \leqq \frac{1}{2}$, except for a simple pole at $s = 0$ with residue

$$\pi^{-1/2}\Gamma\left(1 - \frac{\xi}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\xi}{2}\right),$$

and is bounded as $|s| \to \infty$. Since also $\Phi(s) \to 0$ as $s \to +\infty$ the two sides are analytic continuations of each other and equal

$$\pi^{-1/2}\Gamma\left(1 - \frac{\xi}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\xi}{2}\right)s^{-1}$$

in their respective domains of definition. Therefore Φ is found and the inversion formula for the Mellin transform gives the desired expression for $f(x)$.

Finally we consider case (iv), when the c_j tend to infinity more rapidly than any power of j. In [6] we determined the asymptotic behavior as $N \to \infty$ of the largest eigenvalue (norm) and corresponding eigenfunction of the integral equation

$$\int_0^N k(x+y)f(y)dy = \lambda f(x)$$

under the assumption that k increases rapidly and regularly. Quite analogously one can treat the case of Hankel matrices. We shall only state the result here. Let $\{a_{N,j}\}$ $(0 \leqq j \leqq N)$ be the eigenvector of H_{N+1} corresponding to the largest eigenvalue $\| H_{N+1} \|$, normalized so that

$$a_{N,j} \geqq 0, \quad \sum_{j=0}^{N} a_{N,j}^2 = 1.$$

THEOREM 4.5. *Assume c_j is ultimately increasing and*

$$\frac{c_j}{c_j - c_{j-1}} - \frac{c_{j-1}}{c_{j-1} - c_{j-2}} = o(1) \qquad j \to \infty.$$

Then as $N \to \infty$ we have

$$\| H_{N+1} \| \sim \frac{c_{2N}^2}{c_{2N} - c_{2N-2}},$$

$$\sum_{j=0}^{N} \left[a_{N,j} - \frac{(c_{2N} - c_{2N-2})^{1/2}}{c_{2N}} c_{2j}^{1/2} \right]^2 \to 0.$$

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